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The stability of stratified shear flows

By M. R. COLLYER
Department of Mathematics, University of Essex, Colchester†

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Small perturbations of a parallel shear flow $U(z)$ in an inviscid, incompressible, stably stratified fluid of density $\rho(z)$ are considered, for which the principal measure of stability is the Richardson number, R. For an arbitrary velocity and density profile we discuss the problem of determining whether a curve of neutral stability has adjacent unstable regions in an (α, R) plane, where α is the wave-number of the disturbance. Neutral curves bounding unstable regions are then obtained for a triangular jet flow in conjunction with various density distributions. A comparison is also made between the stability characteristics of jet and shear flows with corresponding density structures.

1. Introduction

The stability of a steady, plane, parallel flow $(U(z), 0, 0)$ in an inviscid, incompressible, stably stratified fluid of Brunt–Väisälä frequency $N(z)$ (where

$$N^2 = -g(d\rho/dz)/\rho$$

with g gravity and $\rho(z)$ the density) is dependent on the Richardson number

$$R = \frac{N^2}{(dU/dz)^2}. \tag{1.1}$$

It was demonstrated by Miles (1961) that a sufficient condition for stability is that $R > \frac{1}{4}$ at all levels. If $R < \frac{1}{4}$ at a certain level then the flow is not necessarily unstable, as the stability characteristics are dependent on the particular velocity and density profile under consideration.

If we represent the vertical velocity w_1 for any infinitesimal disturbance by spatially periodic wave motion so that

$$w_1 = \mathscr{R}\left\{\left(\frac{\rho_0}{\rho}\right)^{\frac{1}{2}} w(z)\, e^{i\alpha(x-ct)}\right\}, \tag{1.2}$$

where ρ_0 is a reference density, α is the wave-number and $c = c_r + ic_i$ is the phase speed, (with instability occurring if $c_i > 0$), then the stability characteristics are determined by the Taylor–Goldstein equation

$$\frac{d^2w}{dz^2} + \left\{\frac{N^2}{(U-c)^2} - \frac{d^2U/dz^2}{U-c} - \alpha^2\right\} w = 0. \tag{1.3}$$

The simplest method of establishing these characteristics, for any particular flow, is to obtain the neutral eigensolutions ($c_i = 0$) of (1.3) satisfying required

† Present address: Royal Aircraft Establishment, Farnborough, Hants.

boundary conditions. It may then be possible to construct a stability boundary (or neutral curve) which is a locus of neutral eigenvalues for which there exist contiguous eigenvalues with $c_i > 0$ in an (α, R) plane. Indeed, neutral curves have been determined for many special velocity and density distributions (see Drazin & Howard 1966), although these have been primarily for examples with monotonic velocity profiles rather than the jet type.

The lack of neutral curves associated with jet flows is only partly due to the difficulty of obtaining neutral eigensolutions to (1.3) for any specified profile, because even if a neutral eigensolution can be found it still has to be resolved whether or not it forms part of a stability boundary. The former problem may be overcome by approximating to the velocity and density distribution with a layered model, in each layer of which both N and dU/dz are treated as constants. This approach has been adopted only for monotonic velocity distributions, with Miles & Howard (1964) using a three-layer model which is effectively an approximation to $U = \tanh z$. The resulting neutral curve bears a close resemblance to that obtained by Drazin (1958) with the $\tanh z$ profile. Therefore, as it has not been possible to construct any neutral curve associated with $U = \operatorname{sech} z$, we examine the stability of a jet flow by considering a four-layer model in which the velocity distribution is defined by a triangular jet profile. However, before we proceed to study this example it is necessary to clarify the other problem of determining whether a neutral eigensolution has contiguous unstable modes in an (α, R) plane.

2. The neutral curve

If there exists a solution to the stability equation (1.3) satisfying the boundary conditions

$$w = 0 \quad \text{at} \quad z = z_1, z_2, \tag{2.1}$$

then the resulting eigenvalue equation may be placed in the form

$$f(\alpha, R, c) = 0, \tag{2.2}$$

and (2.2) possesses the property that it is invariant under complex conjugation. Thus there is stability when c is real and instability for c complex, so that we may represent the solution to the eigenvalue equation as occurring in either one of the two possible states: (I) the (α, R) plane is divided into separate regions in which $c_i = 0$ or $c_i \neq 0$; or (II) $c_i = 0$ for all values of α, R.

If $c_i = 0$ there exists a 'critical level' z_c in the interval (z_1, z_2) at which $U(z_c) = c$, and any neutral eigensolution of (1.3) has a branch point at this level. Hence the solution is multi-valued over part of the range. Miles (1961) has discussed this difficulty and suggested that by including viscosity ν and heat conductivity K, and then determining the asymptotic solution as these parameters tend to zero, (or by defining an initial value problem and then determining its asymptotic solution as $t \to \infty$), that the inviscid solution is the limit of the solution $c_i \to 0+$. The neutral eigensolutions are then defined through this limiting process, and are taken to be single-valued over the entire range of z.

However, this limiting process which Miles defines yields information about *any* neutral eigensolution. But we are principally interested in those neutral eigensolutions which are contiguous to *inviscid* eigensolutions with $c_i > 0$, i.e. the case for which $K, \nu = 0$ and $c_i \to 0$ from a region in the (α, R) plane in which $c_i \neq 0$. It is through this limiting process that a neutral curve (in the sense of § 1) is defined, and to obtain that curve we must consider the limits $K, \nu \to 0$, $c_i \to 0$ *in that order*. We therefore adopt this approach which implies that in general the inviscid neutral eigensolutions are indeterminate. This indeterminancy may be removed, as Miles showed, by defining a limiting process $c_i \to 0$, $K, \nu \to 0$ in that order, but this defines all the neutral eigensolutions whereas the other defines only those which are contiguous to unstable modes in an (α, R) plane.

The analysis of Miles (1961) also showed, with the boundary conditions (2.1), that a necessary condition for any inviscid neutral motion is $Q = 0$, where

$$Q = \int_{z_1}^{z_2} \left(\tau \frac{dU}{dz}\right) dz \tag{2.3}$$

is the rate at which mean wave energy is transferred from the mean flow to the perturbation flow by the Reynolds stress

$$\tau = -\rho\overline{u_1 w_1}, \tag{2.4}$$

and where $(u_1, 0, w_1)$ is the perturbation velocity field with a bar denoting a mean with respect to x. Although $Q = 0$ for all $c_i = 0$, due to the limiting process defined above Q is in general multi-valued and is determinate only in the limit $c_i \to 0$. Hence we have a pair of equations resulting from the eigenvalue problem and the energy condition given by

$$\left.\begin{aligned} f(\alpha, R, c) &= 0 \\ Q(\alpha, R, c) &= 0 \end{aligned}\right\} c_i \to 0+, \tag{2.5}$$

which, if we treat c as a parameter, yields two possible situations: (i) we obtain a curve $g(\alpha, R) = 0$ in the (α, R) plane, which forms a boundary to regions in which $c_i > 0$; or (ii) there exists no solution to (2.5) and hence no neutral curve.

Therefore any curve in the (α, R) plane determined by a solution to (2.5) is necessarily a neutral curve, since it is contiguous to eigenvalues with $c_i > 0$. However we have not proved that this is the entire neutral curve, and also if case (ii) applies the flow is stable for all R. For example, suppose we have a flow such that there exists a region in the (α, R) plane with $c_i = \text{const.}$ $(\neq 0)$, then (2.5) does not yield the neutral curve bounding this region. Thus to complete the proof that (2.5) is sufficient to determine the entire neutral curve, (i.e. there is a one-one mapping from the set (i, ii) to (I, II)), it is necessary to demonstrate, by considering the neutral eigensolution defined in the limit $c_i \to 0+$, $K, \nu \to 0$ and if $c_i \neq 0$ the eigensolution with $c_i > 0$, that c is a continuous function of α for fixed R.

This result has been proved by Miles (1963) if N and U are analytic functions, and it can easily be shown to be true for a model in which N and U are piecewise continuous functions. However Miles (1963) was able to utilize this result only for the case of monotonic velocity profiles, whereas with our approach we may conclude with the aid of this result that (2.5) determines the entire neutral curve for any arbitrary velocity and density profile.

3. Formulation of the jet stability problem

If we choose a Cartesian (x, z) co-ordinate system, so that the velocity U at $z = \pm\infty$ is zero, then we may represent the velocity field for a triangular jet profile as

$$U = \begin{cases} 0 & (|z| \geqslant 1), \\ 1-|z| & (|z| \leqslant 1), \end{cases} \tag{3.1}$$

where the depth of a shear layer and maximum velocity of the jet have been chosen as the respective units of length and velocity. We define the Brunt–Väisälä frequency N by

$$N^2 = \begin{cases} J_+ & (z > 0), \\ J_- & (z < 0), \end{cases} \tag{3.2}$$

with J_+ and J_- constant, so that the Richardson number is given by J_+ and J_- in the upper and lower shear layers respectively, while it is zero in the outer layers.

The solution to (1.3) for the distribution of U and N specified by (3.1) and (3.2) is given by

$$w = \begin{cases} C_+ e^{-\lambda_+ z} & (z > 1), \\ Z_+^{\frac{1}{2}}[A_+ I_{\nu_+}(\alpha Z_+) + B_+ I_{-\nu_+}(\alpha Z_+)] & (0 < z < 1), \\ Z_-^{\frac{1}{2}}[A_- I_{\nu_-}(\alpha Z_-) + B_- I_{-\nu_-}(\alpha Z_-)] & (-1 < z < 0), \\ C_- e^{+\lambda_- z} & (z < -1), \end{cases} \tag{3.3}$$

where
$$Z_\pm = (1-c) \mp z, \tag{3.4a, b}$$

I_ν is a modified Bessel function, and $\nu_\pm$ is defined by

$$\nu_\pm = (\tfrac{1}{4} - J_\pm)^{\frac{1}{2}}, \tag{3.5a, b}$$

while if $J_\pm > \frac{1}{4}$ then we replace $\nu_\pm$ by $i\mu_\pm$. We have determined the solution (3.3) subject to the boundary conditions at $z = \pm\infty$ and $\lambda_\pm$ (which must necessarily be real in order to satisfy these conditions) is given by

$$\lambda_\pm = \frac{1}{c}[(\alpha c)^2 - J_\pm]^{\frac{1}{2}}. \tag{3.6a, b}$$

At an interface between layers the boundary conditions require that the displacement and pressure are continuous. Consequently

$$\text{(i) } w \text{ is continuous} \tag{3.7}$$

and
$$\text{(ii) } \frac{dw}{dz} - \frac{dU/dz}{(U-c)} w \text{ is continuous} \tag{3.8}$$

at an interface. By applying (3.7) and (3.8) at $z = \pm 1$ we obtain, after some straightforward algebraic calculations,

$$A_\pm \phi_\pm^{(1)}(-r) + B_\pm \phi_\pm^{(2)}(-r) = 0, \tag{3.9a, b}$$

where
$$\phi_\pm^{(1)}(r) = [\tfrac{1}{2} - (r^2 - J_\pm)^{\frac{1}{2}}] I_{\nu_\pm}(r) - r I'_{\nu_\pm}(r), \tag{3.10a, b}$$

$$\phi_\pm^{(2)}(r) = [\tfrac{1}{2} - (r^2 - J_\pm)^{\frac{1}{2}}] I_{-\nu_\pm}(r) - r I'_{-\nu_\pm}(r), \tag{3.11a, b}$$

and
$$r = \alpha c. \tag{3.12}$$

At $z = 0$ the boundary conditions yield

$$A_+ I_{\nu_+}(s) + B_+ I_{-\nu_+}(s) - A_- I_{\nu_-}(s) - B_- I_{-\nu_-}(s) = 0, \tag{3.13}$$

and
$$A_+ G_{\nu_+}(s) + B_+ G_{-\nu_+}(s) + A_- G_{\nu_-}(s) + B_- G_{-\nu_-}(s) = 0, \tag{3.14}$$

where
$$s = \alpha(1-c), \tag{3.15}$$

and the function G_ν is defined as

$$G_{\pm\nu}(s) = \tfrac{1}{2} I_{\pm\nu}(s) - s I'_{\pm\nu}(s). \tag{3.16}$$

In the above set of equations the dashes refer to derivatives with respect to the variable in the argument of the Bessel function, i.e.

$$I'_\nu(\theta) = \frac{d}{d\theta}\{I_\nu(\theta)\}. \tag{3.17}$$

	Value of $\alpha\tau/\rho_0$	Range of z
$J_+ < \frac{1}{4}$	0	$z_c < z < 1$
	$\dfrac{\lvert A_+\rvert^2}{\pi}\dfrac{\phi_+^{(1)}(r)}{\phi_+^{(2)}(r)}\sin \nu_+\pi \sin 2\nu_+\pi$	$0 < z < z_c$
$J_- < \frac{1}{4}$	$-\dfrac{\lvert A_-\rvert^2}{\pi}\dfrac{\phi_-^{(1)}(r)}{\phi_-^{(2)}(r)}\sin \nu_-\pi \sin 2\nu_-\pi$	$-z_c < z < 0$
	0	$-1 < z < -z_c$
$J_+ > \frac{1}{4}$	0	$z_c < z < 1$
	$\dfrac{\lvert A_+\rvert^2}{4\pi}(e^{\mu_+\pi} - e^{-\mu_+\pi})(e^{4\mu_+\pi} - 1)$	$0 < z < z_c$
$J_- > \frac{1}{4}$	$-\dfrac{\lvert A_-\rvert^2}{4\pi}(e^{\mu_-\pi} - e^{-\mu_-\pi})(e^{4\mu_-\pi} - 1)$	$-z_c < z < 0$
	0	$-1 < z < -z_c$

TABLE 1. The value of $\tau_\pm$

The other equation which is needed to construct the neutral curve is the energy condition $Q = 0$, which from (2.3) is

$$\int_{-\infty}^{\infty}\left(\tau\frac{dU}{dz}\right)dz = 0, \tag{3.18}$$

with the Reynolds stress τ determined in the limit $c_i \to 0+$. For the case of neutral motion, from Miles (1961), τ is given by

$$\tau = \frac{\rho_0}{2\alpha}\mathscr{I}\left(w^*\frac{dw}{dz}\right), \tag{3.19}$$

where the asterisk denotes a complex conjugate. If $c_i = 0$ then $Z_\pm$ changes sign at the critical levels $\pm z_c$, where $z_c = (1-c)$. In the upper shear layer, for $0 < z < z_c$, $Z_+ > 0$ and thus $I_{\pm\nu_+}(Z_+)$ is real and positive. But $Z_+ < 0$ for $z_c < z < 1$, so that

$$I_\nu(-\bar{Z}_+) = e^{\pm i\nu\pi} I_\nu(\bar{Z}_+), \tag{3.20}$$

where $Z_+ = -\bar{Z}_+$. A choice of sign exists in (3.20) and as discussed in §2 this can be resolved only in the limit $c_i \to 0$. Since $\mathscr{I}(Z_+) < 0$ for $c_i > 0$ we accordingly restrict $Z_\pm$ to the lower half of the complex plane, which implies that in the limit $c_i \to 0+$ we must choose the negative sign of the exponent in (3.20). With this restriction, and a similar one for the lower shear layer, w is a well-defined single-valued function. This enables w^* and dw/dz to be calculated from (3.3) and substituted into (3.19), and through (3.9a, b) we may express τ in terms of a single arbitrary constant. The results of these straightforward algebraic calculations for the various cases are summarized in table 1.

From table 1 it can be seen that τ is constant, and irrespective of the value of J it is non-zero only in the interval $(-z_c, z_c)$. Since $dU/dz = 0$ for $|z| > 1$, (3.18) simplifies to

$$\tau_+ - \tau_- = 0, \tag{3.21}$$

where the values of τ_+ and τ_- are given in table 1. In one special case, when $J_+ = J_- = J(<\frac{1}{4})$, (3.21) is greatly simplified so that the neutral curve can easily be constructed. We shall examine this example in detail in the next section, and it provides a basis for a discussion of the more complicated situation in which $J_+ > \frac{1}{4}$ and $J_- < \frac{1}{4}$.

4. $J_+ = J_- = J(<\frac{1}{4})$

In this section we examine an atmosphere of constant Brunt–Väisälä frequency. Because of the symmetry one of the equations, (3.21), which determines the neutral curve has only the trivial solution $A_+ = A_- = 0$ or $\phi_+^{(1)} = \phi_-^{(1)} = 0$, which implies through (3.9a, b) that $B_+ = B_- = 0$. (Thus $\tau = 0$ for all z, i.e. there is no exchange of mean wave energy between the mean flow and the perturbation flow at any level.)

If we consider the eigenvalue problem in the limit $c_i \to 0+$ and put $B_+ = B_- = 0$ then (3.9a) and (3.9b) degenerate into a single equation:

$$[\tfrac{1}{2} - (r^2 - J)^{\frac{1}{2}}]\, I_\nu(r) - r I'_\nu(r) = 0, \tag{4.1}$$

while (3.13) reduces to an identity and (3.14) becomes $G_\nu(s) = 0$, i.e.

$$\tfrac{1}{2} I_\nu(s) - s I'_\nu(s) = 0. \tag{4.2}$$

Thus the neutral curve is determined by the solutions to the pair of equations (4.1) and (4.2) with α and c given by

$$\left.\begin{aligned} \alpha &= r + s \\ c &= r/(r+s) \end{aligned}\right\} \quad r > J^{\frac{1}{2}},\ s > 0, \tag{4.3}$$

where use has been made of (3.12) and (3.15).

Miles & Howard (1964) have discussed an equation of similar type to (4.1) and from their analysis we may deduce that (4.1) and (4.2) each have one and only one solution for fixed ν in the interval $(-\frac{1}{2}, \frac{1}{2})$. Therefore there exists only one value of α corresponding to a point on the neutral curve in the (α, J) plane for each ν. A programme was written to evaluate these solutions for particular values of ν, and the resulting values of α and c are used to plot a neutral curve in the (α, J) plane, which is shown in figure 1. At $J = \frac{1}{4}$ only a disturbance with a wave-number of 1·663 and a wave speed of 0·360 is 'critically' stable in that it is contiguous to unstable modes. But as J decreases so the waveband of unstable wavelengths increases, until at $J = 0$ all wave-numbers such that $0 < \alpha < 1{\cdot}838$ are unstable for certain wave speeds. The homogeneous problem ($J = 0$) has been investigated by Rayleigh (1880) and the range of unstable wavelengths is in agreement with his results.

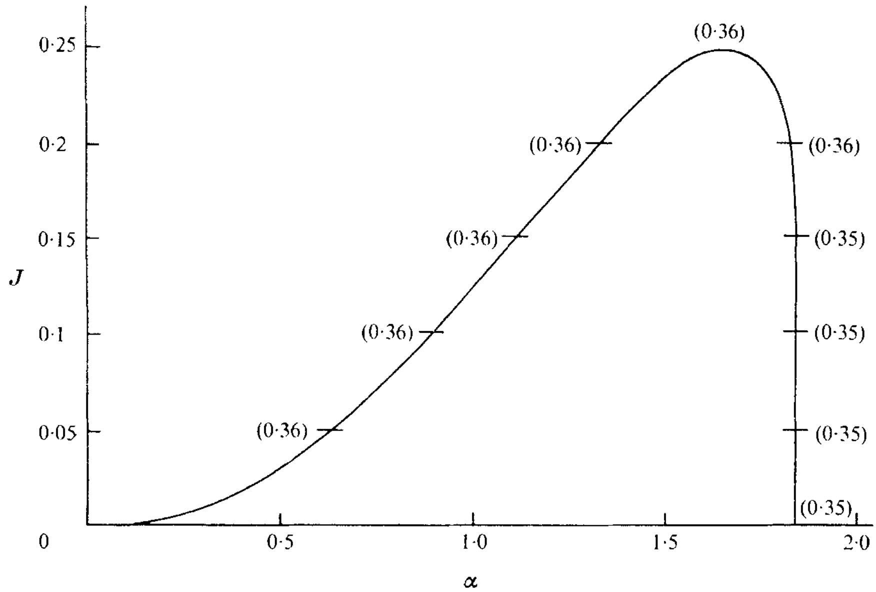

FIGURE 1. The neutral curve for jet flow with N constant throughout the atmosphere. Numbers in brackets denote the wave speed.

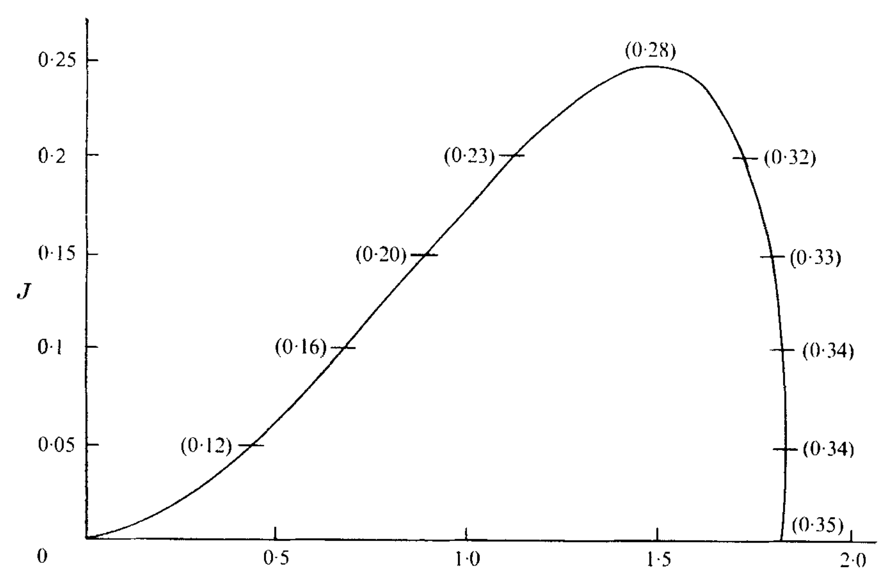

FIGURE 2. The neutral curve for jet flow with N zero in the outer layers. Numbers in brackets denote the wave speed.

To complete this section we consider a model with constant density ($N = 0$ in the outer layers, since in these circumstances the boundary conditions at $z = \pm\infty$ imply no restrictions on the small wave-numbers as acceptable modes. The only effect this has on the earlier analysis is to alter (4.1), (which is dependent on the conditions in the outer layers), to

$$(\tfrac{1}{2}-r)\,I_\nu(r) - rI'_\nu(r) = 0. \tag{4.4}$$

Thus (4.4) and (4.2) are the pair of equations for α and c which determine the neutral curve. (α and c are defined as in (4.3) but with $r > 0$.) These equations differ only slightly from the previous pair, so that by following a similar procedure as before we obtain the neutral curve which is plotted in figure 2.

For $J = 0$ we know that the unstable waveband remains unaltered, but the shape of the entire neutral curve is very similar to figure 1 with the critical wave-number at $J = \frac{1}{4}$ only slightly reduced to 1·481 for a wave speed of 0·280. The main difference between the examples is that in the former one the wave speed is nearly constant at all points along the neutral curve, while for the latter it varies with α and J.

5. The stability of an isolated shear layer

We digress in this section from the main problem to examine the stability of a shear layer embedded between layers of constant velocity, which was originally studied by Goldstein (1931) and subsequently by Miles & Howard (1964). In order to facilitate a comparison with the jet profile the velocity field is defined as

$$U = \left\{\begin{array}{ll} 1 & (z > 0), \\ 1+z & (-1 < z < 0), \\ 0 & (z < -1), \end{array}\right\} \tag{5.1}$$

while the Brunt–Väisälä frequency is defined by (3.2).

The solution to the stability equation (1.3) in the interval $(-1, 0)$ is thus unaltered from the jet case, and through a result of Miles (1961) we can establish that the energy condition $Q = 0$ restricts this solution so that either $A_- = 0$ or $B_- = 0$. With this restraint the equation resulting from the application of the boundary conditions at $z = -1$ is

$$[\tfrac{1}{2} - (r^2 - J_-)^{\frac{1}{2}}]\,I_{\nu_-}(r) - rI'_{\nu_-}(r) = 0. \tag{5.2}$$

From the symmetry of the situation the boundary conditions at $z = 0$ yield

$$[\tfrac{1}{2} - (s^2 - J_+)^{\frac{1}{2}}]\,I_{\nu_-}(s) - sI'_{\nu_-}(s) = 0, \tag{5.3}$$

which with (5.2) determines the entire neutral curve if

$$\left.\begin{array}{l} \alpha = r+s \\ c = r/(r+s) \end{array}\right\} r > J_-^{\frac{1}{2}}, \quad s > J_+^{\frac{1}{2}}. \tag{5.4}$$

For an atmosphere in which $J_+ = J_- = J$ the pair of equations (5.2) and (5.3) are identical, so that $r = s$, i.e. $c = \frac{1}{2}$ at all points of the neutral curve. As this equation is also identical to (4.1) there is a simple connexion between the neutral curves for the two velocity profiles. It can be stated as: In an atmosphere of constant Brunt–Väisälä frequency the neutral curves for the jet profile (3.1) and the shear profile (5.1) are linked by the relationship

$$\alpha_j c_j = \tfrac{1}{2}\alpha_s \tag{5.5}$$

for any fixed value of ν, where the subscripts j and s refer to the jet and shear flow values respectively.

With the aid of (5.5) we may deduce from figure 1 the neutral curve for shear flow in an atmosphere of constant Brunt–Väisälä frequency. The resulting curve

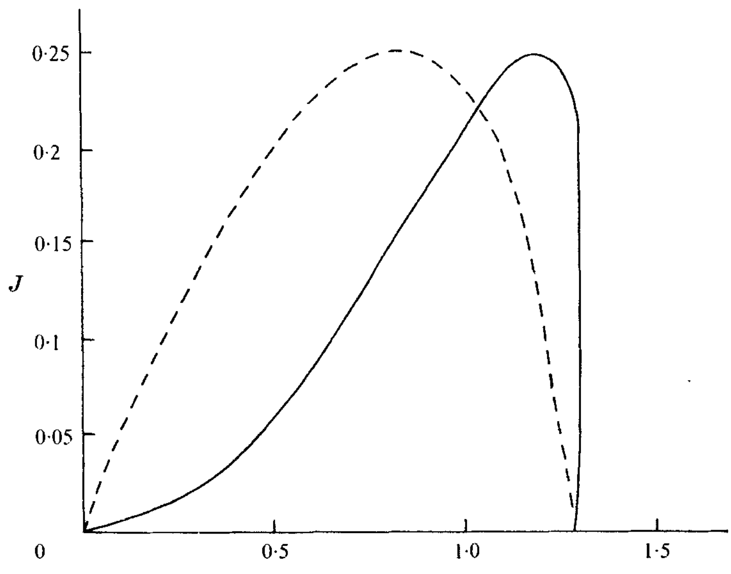

FIGURE 3. Neutral curves for shear flow. ——, N constant throughout atmosphere; - - - -, N zero in outer layers (from Miles & Howard 1964).

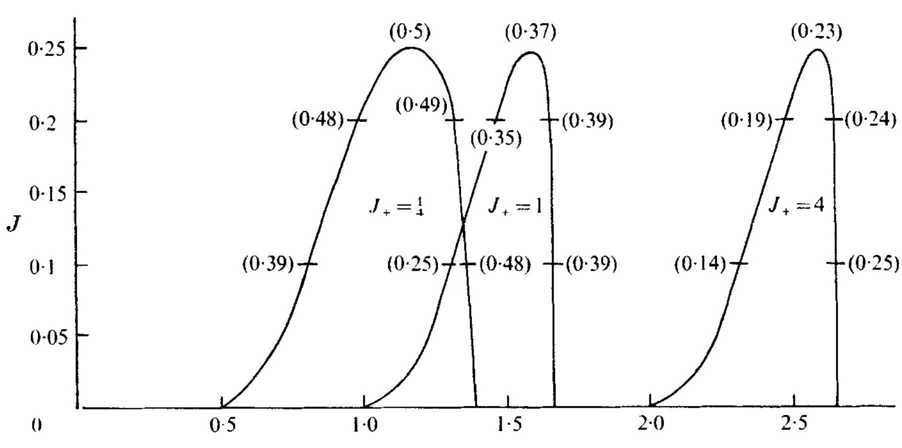

FIGURE 4. Neutral curves for shear flow in an (α, J_-) plane for three values of J_+: $\frac{1}{4}$, 1 and 4. Numbers in brackets denote the wave speed.

is sketched in figure 3, together with Miles & Howard's neutral curve for the case where $N = 0$ in the outer layers. By a similar argument to the one above this could have been determined from the curve shown in figure 2.

Finally, we revert to the problem posed by (5.2) and (5.3) and examine the effect on the stability of different values of N in the uppermost layer. The restriction that $s > J_+^{\frac{1}{2}}$ prevents the small wave-numbers from being acceptable modes. However as J_+ increases so the solution to (5.3) tends rapidly towards J_+; for example $(s - J_+^{\frac{1}{2}}) < 10^{-3}$ when $J_+ = 4$. Hence the shape of the neutral curve in an (α, J_-) plane is determined effectively by the solutions to (5.2) and the absolute value by the stability parameter N in the uppermost layer. This is demonstrated in figure 4 by neutral curves for various values of J_+.

6. $J_+ > \frac{1}{4}$ and $J_- < \frac{1}{4}$

We return to the problem proposed at the end of §3 on the stability of a jet stream in an atmosphere for which $J_+ > \frac{1}{4}$ and $J_- < \frac{1}{4}$. In the symmetrical case, discussed in §4, r and s could be determined separately whereas for this case they are interlinked so that the position is extremely complicated. Therefore it will be more profitable to discuss the implications of the energy condition (3.21) alone, and from this derive a few general results.

From table 1 the energy condition is given by

$$|A_+|^2 (e^{\mu_+\pi} - e^{-\mu_+\pi}) (e^{4\mu_+\pi} - 1) = -4|A_-|^2 \frac{\phi_-^{(1)}(r)}{\phi_-^{(2)}(r)} \sin \nu_-\pi \sin 2\nu_-\pi. \tag{6.1}$$

Since $\mu_+ > 0$, the left-hand side of (6.1) is always positive, so that a solution to (6.1) exists only if

$$\frac{\phi_-^{(1)}(r)}{\phi_-^{(2)}(r)} < 0. \tag{6.2}$$

This result implies that both $\tau_+ > 0$ and $\tau_- > 0$ for neutrally stable waves which are contiguous to unstable waves in an (α, R) plane. Hence this is an example of a situation where mean wave energy can be transferred between the mean flow and the perturbation flow. It always takes place in such a way that in the lower shear layer, where $J_- < \frac{1}{4}$, the transfer is from the mean flow to the perturbation flow, while in the upper shear layer it is in the reverse direction. However $\tau_- \to 0$ as $J_- \to \frac{1}{4}$, i.e. the amount of energy transferred tends to zero.

In order to examine condition (6.2) we first note that under a transformation $\nu_- \to -\nu_-$, $\phi_-^{(1)} \to \phi_-^{(2)}$ and $\phi_-^{(2)} \to \phi_-^{(1)}$, so that it is sufficient to consider ν_- in the interval $(0, \frac{1}{2})$. We also need to refer to certain results used in §4. In particular, as the equation $\phi_-^{(1)}(r) = 0$ is identical in form to (4.1) we can show that it has one and only one solution for fixed ν_-, say at $r = r_1$. Similary $\phi_-^{(2)}(r) = 0$ has only one solution, at $r = r_2$ say. Thus from (3.10*b*) and (3.11*b*) we have that

$$\begin{aligned} \phi_-^{(2)}(r_1) &= \frac{r_1 I'_{\nu_-}(r_1)}{I_{\nu_-}(r_1)} I_{-\nu_-}(r_1) - r_1 I'_{-\nu_-}(r_1) \\ &= \frac{2 \sin \nu_- \pi}{\pi I_{\nu_-}(r_1)} > 0, \end{aligned} \tag{6.3}$$

and similarly
$$\phi_-^{(1)}(r_2) = -\frac{2\sin\nu_-\pi}{\pi I_{-\nu_-}(r_2)} < 0. \quad (6.4)$$

Furthermore, for fixed values of ν_-, both $\phi_-^{(1)}(r)$ and $\phi_-^{(2)}(r)$ are monotonic decreasing functions of r. Therefore with the aid of (6.3) and (6.4) we can prove that $r_2 > r_1$, and there exists only a finite interval (r_1, r_2) in which $\phi_-^{(1)}/\phi_-^{(2)}$ is negative and condition (6.2) is satisfied.

Thus (6.2) is a necessary (but not sufficient) condition for instability, and is independent of the value of J_+. We can therefore deduce if $J_+ > r_2^2$ that the flow is stable, since the boundary condition at $z = +\infty$ is satisfied only if $r > J_+^{\frac{1}{2}}$. From our analysis in §4 we may obtain the value of r_2 for any particular value of J_-; for example, if $J_- = 0{\cdot}15$ then $r_2 = 0{\cdot}655$ and the flow is stable if $J_+ > 0{\cdot}429$. The possibility that no unstable waves develop for certain values of N in the uppermost layer appears to be the principal distinction between the shear flow and jet flow cases.

In conclusion, it is evident for any combination of velocity and density profiles studied that the flow is unstable if $R < \frac{1}{4}$ at any level, unless the unstable wavelengths cannot be excited due to restrictions imposed by the boundary conditions.

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